BZ^3

```
> restart:
> with(Involutive): with(homalg):
```

Tell `Involutive` to compute over the integers:

```
> InvolutiveOptions("rational",false);
```

$$true$$

```
> RPI:=`Involutive/homalg`;
```

$$RPI := Involutive/homalg$$

```
> `homalg/default`:=RPI;
```

$$homalg/default := Involutive/homalg$$

The infinite cyclic group $\mathbb{Z}^3 := (\mathbb{Z}^3, +) = \underbrace{C_\infty \times \cdots \times C_\infty}_{3}$:

The group ring $\mathbb{Z}\mathbb{Z}^3 = \mathbb{Z}[x, x^{-1}, y, y^{-1}, a, a^{-1}]$:

```
> var:=[[x,y,a,X,Y,A],[x*X-1,y*Y-1,a*A-1]];
```

$$var := [[x,\ y,\ a,\ X,\ Y,\ A],\ [x\,X-1,\ y\,Y-1,\ a\,A-1]]$$

The trivial $\mathbb{Z}\mathbb{Z}^3$-module $\mathbb{Z}$:

```
> Z:=[x-1,y-1,a-1];
```

$$Z := [x-1,\ y-1,\ a-1]$$

Theorem: *For a discrete group G there are natural isomorphisms*

$$\mathrm{Ext}^\bullet_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) = \mathrm{H}^\bullet(G, \mathbb{Z}) = \mathrm{H}^\bullet(\mathrm{B}G; \mathbb{Z}).$$

The integral cohomology of the group $\mathbb{Z}^n$ = integral cohomology of the classifing space

$$\mathrm{B}Z = (S^1)^n = T^n,$$

the n-dimensional torus.

Theorem: *The cohomology ring of $\mathbb{Z}^n$ is* $\mathrm{Ext}^\bullet_{\mathbb{Z}\mathbb{Z}^n}(\mathbb{Z}, \mathbb{Z}) = \mathrm{H}^\bullet(\mathrm{B}\mathbb{Z}^n; \mathbb{Z}) = \mathrm{H}^\bullet(T^n; \mathbb{Z}) = \Lambda\mathbb{Z}^n$.

The zeroth integral cohomology $\mathrm{H}^0(T^3; \mathbb{Z}) = \mathbb{Z}$:

```
> Ext(0,Z,Z,var);
```

$$[[1 = \begin{bmatrix} 1 \end{bmatrix}],\ [A-1,\ Y-1,\ X-1,\ a-1,\ y-1,\ x-1],\ \text{“Presentation”}]$$

The first integral cohomology $\mathrm{H}^1(T^3; \mathbb{Z}) = \mathbb{Z}^3$:

```
> Ext(1,Z,Z,var);
```

$$\left[\left[[1,\ 0,\ 0] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ [0,\ 1,\ 0] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ [0,\ 0,\ 1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right],\ [[0,\ 0,\ A-1],\ [0,\ A-1,\ 0],\ [A-1,\ 0,\ 0],\right.$$
$$[0,\ 0,\ Y-1],\ [0,\ Y-1,\ 0],\ [Y-1,\ 0,\ 0],\ [0,\ 0,\ X-1],\ [0,\ X-1,\ 0],\ [X-1,\ 0,\ 0],$$
$$[0,\ 0,\ a-1],\ [0,\ a-1,\ 0],\ [a-1,\ 0,\ 0],\ [0,\ 0,\ y-1],\ [0,\ y-1,\ 0],\ [y-1,\ 0,\ 0],$$
$$\left.[0,\ 0,\ x-1],\ [0,\ x-1,\ 0],\ [x-1,\ 0,\ 0]],\ \text{“Presentation”}\right]$$

The second integral cohomology $\mathrm{H}^2(T^3; \mathbb{Z}) = \mathbb{Z}^3$:

```
> Ext(2,Z,Z,var);
```

$$\left[\left[[1,\,0,\,0] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\,[0,\,1,\,0] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\,[0,\,0,\,1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right.,\,[[0,\,0,\,A-1],\,[0,\,A-1,\,0],\,[A-1,\,0,\,0],$$
$$[0,\,0,\,Y-1],\,[0,\,Y-1,\,0],\,[Y-1,\,0,\,0],\,[0,\,0,\,X-1],\,[0,\,X-1,\,0],\,[X-1,\,0,\,0],$$
$$[0,\,0,\,a-1],\,[0,\,a-1,\,0],\,[a-1,\,0,\,0],\,[0,\,0,\,y-1],\,[0,\,y-1,\,0],\,[y-1,\,0,\,0],$$
$$\left.[0,\,0,\,x-1],\,[0,\,x-1,\,0],\,[x-1,\,0,\,0]],\,\text{“Presentation”}\right]$$

The third integral cohomology $\mathrm{H}^3(T^3;\mathbb{Z}) = \mathbb{Z}$:

```
>   Ext(3,Z,Z,var);
```

$$[[1 = \begin{bmatrix} 1 \end{bmatrix}],\,[A-1,\,Y-1,\,X-1,\,a-1,\,y-1,\,x-1],\,\text{“Presentation”}]$$

The fourth integral cohomology $\mathrm{H}^4(T^3;\mathbb{Z}) = 0$:

```
>   Ext(4,Z,Z,var);
```

$$[[1 = \begin{bmatrix} 0 \end{bmatrix}],\,[1],\,\text{“Presentation”}]$$

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References

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